

# Generalizations of Functionally Generated Portfolios with Applications to Statistical Arbitrage

Winslow Strong\*

ETH Zürich, Department of Mathematics  
CH-8092 Zürich, Switzerland  
winslow.strong@math.ethz.ch

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## Abstract

The theory of functionally generated portfolios (FGPs) is an aspect of the continuous-time, continuous-path Stochastic Portfolio Theory of Robert Fernholz. FGPs have been formulated to yield a *master equation* - a description of their return relative to a passive (buy-and-hold) benchmark portfolio serving as the numéraire. This description has proven to be analytically very useful, as it is both pathwise and free of stochastic integrals. Historically, FGPs have been specified only as portfolios on the tradeable assets of a market, and the numéraire has been confined to be the market portfolio. Here we generalize the class of FGPs in several ways: (1) they may be specified over any strictly positive wealth processes resulting from investment in the tradeable assets, (2) the numéraire may be any strictly positive wealth process, (3) generating functions may be stochastically dynamic, adjusting to changing market conditions through an auxiliary continuous-path stochastic argument of finite variation. These generalizations do not forfeit the important tractability properties of the associated master equation. We show how these generalizations can be usefully applied to statistical arbitrage, portfolio risk immunization, and the theory of mirror portfolios.

**Keywords:** Stochastic Portfolio Theory, functionally generated portfolio, statistical arbitrage, portfolio theory, portfolio immunization, mirror portfolio, master equation

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# 1 Introduction and background

Functionally generated portfolios (FGPs) were introduced by Robert Fernholz in [6, 8], see also [9, 11]. They have historically been constructed by selecting a deterministic generating function that takes the market portfolio as its argument. They are notable for admitting a description of their performance, relative to a passive (buy-and-hold) numéraire, that is both pathwise and free of stochastic integrals. This description is known as the *master equation*, and is a useful tool for portfolio analysis and optimization.

In markets that are uniformly elliptic and diverse [12], and more generally those markets with sufficient intrinsic volatility [10], FGPs yield explicit portfolios that are arbitrages relative to the market portfolio (although see [16] for an alternative diverse market model that is compatible with no-arbitrage). In more general equity market models, FGPs are useful for exploiting certain statistical regularities, such as the stability of the distribution of capital over time [9, 11], and the non-constancy of the rate of variance of log-prices as a function of sampling interval [5]. These FGP-derived portfolios are best described as *statistical arbitrages* [5], since they exploit the aforementioned statistical regularities in the data to achieve favorable risk-return profiles. One of the main attractions of the techniques presented in this paper will undoubtedly be towards characterizing and optimizing such statistical arbitrage portfolios.

This paper is organized as follows. In Section 2 a market model is laid out, typical of those used in Stochastic Portfolio Theory, and the notion of a *synthetic market* is introduced. Synthetic markets consist of strictly positive wealth processes of arbitrary portfolios invested in the tradeable assets. In Section 3 the class of FGPs is extended from its historical definition in the following ways:

- FGPs may be constructed on synthetic markets. For example, the synthetic assets might be exchange-traded funds (ETFs) composed of stocks rather than the underlying stocks themselves. However, they need not be directly traded on the market as ETFs are.
- An FGP's corresponding master equation may use an *arbitrary* strictly positive wealth process as numéraire, rather than being restricted to the market portfolio or a more general passive portfolio.
- Generating functions may be extended to accommodate continuous-path *auxiliary stochastic arguments* of finite variation, adding greater flexibility to FGPs to accommodate changing market conditions.

In Section 4, some applications are explored showing how the extensions of FGPs are useful for practical problems in portfolio construction. The idea of applying FGPs for statistical arbitrage purposes was originally presented in [5]. In Section 4.1 we generalize and formalize this idea using our more general notion of FGPs. In Section 4.2 we present a method to immunize a given FGP from exposure to a direction of risk in the market that may vary stochastically as a finite variation process. In Section 4.3 we extend the notion of mirror portfolios introduced in [12] and give a result (Corollary 4.6) on their asymptotic properties: under realistic market conditions either a portfolio, its mirror, or both lose all their value asymptotically. Section 5 summarizes the results, and poses some remaining challenges to tackle for the theory of FGPs.

## 2 Setting and definitions

The market consists of tradeable assets with prices  $X_t = (X_{1,t}, \dots, X_{n,t})'$ , one of which may be a money market account. Most of our analysis here will take place on log prices  $L_t := \log X_t$ . The dynamics of  $L$  are given by

$$dL_t = \gamma_t dt + \sigma_t dW_t,$$

where

$$\sum_{i=1}^n \int_0^t (|\gamma_{i,s}| + a_{ii,s}) ds < \infty, \quad \forall t \geq 0,$$

with  $a_{ij,t} := [\sigma_t \sigma_t']_{ij} = \frac{d}{dt} \langle \log X_i, \log X_j \rangle, \quad 1 \leq i, j \leq n,$

with  $\sigma_t$  taking values in  $\mathbb{R}^{n \times d}$ ,  $d \geq n$ ,  $\forall t \geq 0$ . Throughout, all equalities hold merely almost surely. The process  $\mathbf{1} := (1, \dots, 1)$ , where the dimensionality should be clear from the context.

**Definition 2.1.** A *portfolio*  $\pi$  on  $X$  is a progressively measurable  $\mathbb{R}^n$ -valued process satisfying

$$\int_0^t (|b_{\pi,s}| + \pi_s' a_s \pi_s) ds < \infty, \quad \forall t \geq 0, \tag{2.1}$$

and  $\sum_{i=1}^n \pi_{i,t} = 1, \quad \forall t \geq 0.$

The *wealth process*  $V^{v,\pi}$  arising from investment according to  $\pi$  is given by

$$d \log V_t^{v,\pi} = \gamma_{\pi,t} dt + \sigma_{\pi,t} dW_t,$$

$$V_0^{v,\pi} = v \in (0, \infty),$$

where  $\sigma_\pi := \pi' \sigma$ ,

$$\gamma_{\pi,t} := \pi_t' \gamma_t + \gamma_{\pi,t}^*,$$

$$\text{and } \gamma_{\pi,t}^* := \frac{1}{2} \left( \sum_{i=1}^n \pi_{i,t} a_{ii,t} - \pi_t' a_t \pi_t \right).$$

The process  $\gamma_\pi^*$  is called the *excess growth rate*, and plays an important role in Stochastic Portfolio Theory [9, 11]. To ease notation, we will use  $V_t^\pi := V_t^{1,\pi}$  and often omit the subscript “ $t$ ” when referring to processes, in general.

The prior study of portfolio generating functions [6, 8] required the *market portfolio*  $\mu$  to be both the argument to the generating function and the numéraire for wealth. Here we extend the notion of FGPs to a *synthetic market*

$$\mathfrak{X} := (\mathfrak{X}_1, \dots, \mathfrak{X}_m)' = (V^{v_1, \psi^1}, \dots, V^{v_m, \psi^m})',$$

where  $\psi^1, \dots, \psi^m$  are arbitrary portfolios on  $X$  and  $v \in (0, \infty)^m$ . We will typically work with the log

processes  $\mathfrak{L} := \log \mathfrak{X}$ , which satisfy

$$d\mathfrak{L}_t = \mathfrak{g}_t dt + \mathfrak{s}_t dW_t,$$

where  $\mathfrak{g}_j := \gamma_{\psi^j}$ ,

and  $\mathfrak{s}_{j\nu} := [\sigma_{\psi^j}]_\nu$ , for  $1 \leq j \leq m$ ,  $1 \leq \nu \leq d$ .

A *portfolio on  $\mathfrak{X}$*  is a process  $\mathfrak{p}$  taking values in  $\mathbb{R}^m$  and satisfying

$$\int_0^t (|\mathfrak{b}_{\mathfrak{p},s}| + \mathfrak{p}'_s \mathfrak{a}_s \mathfrak{p}_s) ds < \infty, \quad \forall t \geq 0, \quad (2.2)$$

$$\text{and } \sum_{i=1}^m \mathfrak{p}_{i,s} = 1, \quad \forall t \geq 0,$$

$$\text{where } \mathfrak{a}_{ij} := [\mathfrak{s}\mathfrak{s}']_{ij} = \frac{d}{dt} \langle \log \mathfrak{X}_i, \log \mathfrak{X}_j \rangle, \quad 1 \leq i, j \leq m.$$

The *wealth process*  $V^{v,\mathfrak{p}}$  corresponding to the portfolio  $\mathfrak{p}$  is determined by

$$d \log V_t^{v,\mathfrak{p}} = \mathfrak{g}_{\mathfrak{p},t} dt + \mathfrak{s}_{\mathfrak{p},t} dW_t,$$

$$V_0^{v,\mathfrak{p}} = v \in (0, \infty),$$

where  $\mathfrak{s}_{\mathfrak{p}} := \mathfrak{p}' \mathfrak{s}$ ,

$$\mathfrak{g}_{\mathfrak{p},t} := \mathfrak{p}'_t \mathfrak{g}_t + \mathfrak{g}_{\mathfrak{p},t}^*,$$

$$\mathfrak{g}_{\mathfrak{p},t}^* := \frac{1}{2} \left( \sum_{i=1}^m \mathfrak{p}_{i,t} \mathfrak{a}_{ii,t} - \mathfrak{p}'_t \mathfrak{a}_t \mathfrak{p}_t \right).$$

Any portfolio  $\mathfrak{p}$  on  $\mathfrak{X}$  can be represented as a portfolio  $\pi^{(\mathfrak{p})}$  on  $X$  by

$$\pi^{(\mathfrak{p})} := \left( \sum_{j=1}^m \mathfrak{p}_j \psi_1^j, \dots, \sum_{j=1}^m \mathfrak{p}_j \psi_n^j \right), \quad (2.3)$$

in the sense that  $V^{\mathfrak{p}} = V^{\pi^{(\mathfrak{p})}}$ . To verify that  $\pi^{(\mathfrak{p})}$  is indeed a portfolio on  $X$ ,

$$\sum_{i=1}^n \pi_i^{(\mathfrak{p})} = \sum_{i=1}^n \sum_{j=1}^m \mathfrak{p}_j \psi_i^j = \sum_{j=1}^m \mathfrak{p}_j = 1,$$

and

$$\left( \pi^{(\mathfrak{p})} \right)' a \pi^{(\mathfrak{p})} = \mathfrak{p}' \psi' a \psi \mathfrak{p} = \mathfrak{p}' \mathfrak{a} \mathfrak{p},$$

so integrability condition (2.1) is satisfied for  $\pi^{(\mathfrak{p})}$ .

The reader may now be wondering what is the use of the synthetic market  $\mathfrak{X}$  since any  $\mathfrak{p}$  has a corresponding  $\pi^{(\mathfrak{p})}$  with  $V^{\mathfrak{p}} = V^{\pi^{(\mathfrak{p})}}$ . One reason that synthetic markets are useful is because in some cases it is more natural to specify portfolios on  $\mathfrak{X}$  than on  $X$ . As an example,  $(\psi^j)_{1 \leq j \leq m}$  may be portfolios capturing certain qualities of the market, such as value, growth, momentum, dividends, eigenportfolios of  $a$ , etc. A second reason is that Theorem 3.2 below, which extends FGPs to synthetic markets, expands the class of

portfolios obeying a master equation - a pathwise description of relative performance free from stochastic integrals.

There is no similar gain in generality to considering higher-order synthetic markets composed of wealth processes on synthetic markets, as in  $\mathfrak{X}^{(2)} = (V^{\mathbf{p}^1}, \dots, V^{\mathbf{p}^k})$ , where  $(\mathbf{p}^j)^{1 \leq j \leq k}$  are portfolios on  $\mathfrak{X}$ . This market already is synthetic on  $X$ :  $\mathfrak{X}^{(2)} = (V^{\pi^1}, \dots, V^{\pi^k})$ , where  $\pi^j := \pi^{(\mathbf{p}^j)}$  as in (2.3), hence it is covered by results for synthetic markets. Nevertheless, it may be convenient to name synthetic markets in this fashion.

In practice an investor may be able to trade shares in the wealth processes  $\mathfrak{X}_j = V^{\psi^j}$  directly, as in the case of ETFs, or a synthetic construction might be necessary by trading  $\pi^{(\mathbf{p})}$  on  $X$ . Distinguishing between  $X$  and  $\mathfrak{X}$  can be important if and when transaction costs and liquidity constraints are taken into consideration, as turnover and leverage may be vastly different for  $\mathbf{p}$  and  $\pi^{(\mathbf{p})}$ . Those issues are beyond the scope of this paper.

We adopt the following notation for the synthetic market:

$$\begin{aligned}\mathfrak{X}^{\mathfrak{r}} &:= \frac{\mathfrak{X}}{V^{\mathfrak{r}}}, \\ \mathfrak{a}_{ij}^{\mathfrak{r}} &:= [\mathfrak{s}\mathfrak{s}']_{ij} = \frac{d}{dt} \langle \log \mathfrak{X}_i^{\mathfrak{r}}, \log \mathfrak{X}_j^{\mathfrak{r}} \rangle, \\ &= \mathfrak{a}_{ij} - [\mathfrak{a}\mathfrak{r}]_j - [\mathfrak{a}\mathfrak{r}]_i + \mathfrak{a}_{\mathfrak{r}\mathfrak{r}}, \\ \mathfrak{a}_{\mathbf{p}\mathbf{p}} &:= \mathfrak{s}_{\mathbf{p}}\mathfrak{s}'_{\mathbf{p}} = \mathbf{p}'\mathbf{a}\mathbf{p} = \frac{d}{dt} \langle \log V^{\mathbf{p}}, \log V^{\mathbf{p}} \rangle, \\ \mathfrak{a}_{\mathbf{p}\mathbf{p}}^{\mathfrak{r}} &:= \mathbf{p}'\mathfrak{a}^{\mathfrak{r}}\mathbf{p} = \frac{d}{dt} \left\langle \log \left( \frac{V^{\mathbf{p}}}{V^{\mathfrak{r}}} \right), \log \left( \frac{V^{\mathbf{p}}}{V^{\mathfrak{r}}} \right) \right\rangle = \mathfrak{a}_{\mathfrak{r}\mathfrak{r}}^{\mathbf{p}}.\end{aligned}\tag{2.4}$$

The *numéraire invariance property* of  $\mathfrak{g}_{\mathbf{p}}^*$  holds for arbitrary portfolios  $\mathbf{p}$  and  $\mathfrak{r}$ :

$$\mathfrak{g}_{\mathbf{p}}^* = \frac{1}{2} \left( \sum_{i=1}^m \mathbf{p}_i \mathfrak{a}_{ii}^{\mathfrak{r}} - \mathfrak{a}_{\mathbf{p}\mathbf{p}}^{\mathfrak{r}} \right),\tag{2.5}$$

and in particular, since  $\mathfrak{a}_{\mathfrak{r}\mathfrak{r}}^{\mathfrak{r}} = 0$  by (2.4), we have

$$\mathfrak{g}_{\mathfrak{r}}^* = \frac{1}{2} \sum_{i=1}^m \mathfrak{r}_i \mathfrak{a}_{ii}^{\mathfrak{r}}.\tag{2.6}$$

Equation (2.5) can be seen from (2.4) and a bit of algebra: for any square matrix  $c$  of dimensions  $p \times p$  and any vectors  $x$  and  $y$  of dimension  $p$  such that  $\sum_{i=1}^p x_i = \sum_{i=1}^p y_i = 1$ , we have

$$\begin{aligned}& \sum_{i=1}^p x_i (c_{ii} - [y'c]_i - [cy]_i + y'cy) - \sum_{i,j=1}^p x_i \left( c_{ij} - [cy]_j - [cy]_i + y'cy \right) x_j \\ &= \sum_{i=1}^p x_i c_{ii} - 2y'cx + y'cy - (x'cx - 2y'cx + y'cy), \\ &= \sum_{i=1}^p x_i c_{ii} - x'cx.\end{aligned}$$

When  $\mathfrak{r}$  is exclusively invested in the money market, then discounted quantities will be denoted by  $\hat{\mathfrak{X}} := \mathfrak{X}^{\mathfrak{r}}$ ,  $\hat{V}^{\mathfrak{p}} := V^{\mathfrak{p}}/\mathfrak{X}^{\mathfrak{r}}$ , etc. Note also that in this case  $\mathfrak{a}^{\mathfrak{r}} = \mathfrak{a}$ .

### 3 Generalizations of functionally generated portfolios

#### 3.1 A master equation when the investable assets and numéraire are arbitrary wealth processes

Functionally generated portfolios were first introduced in [6, 8], see also [9, 11].

**Definition 3.1.** A *generating function* is a function  $H \in \mathcal{C}^2(\mathbb{R}^m, \mathbb{R})$ .

With respect to the historical work on portfolio generating functions, we formulate them here in the log sense with logarithmic argument. Specifically, our  $H$  is related to the previous notion of generating function  $G$  by  $H(y) = \log G(e^y)$ . This makes the analysis cleaner for our purposes.

**Theorem 3.2.** For generating function  $H$ , synthetic market  $\mathfrak{X}$ , and arbitrary portfolio  $\mathfrak{r}$  on  $\mathfrak{X}$ , let

$$\mathfrak{p} = \lambda \mathfrak{r} + \nabla H(\mathfrak{L}^{\mathfrak{r}}), \quad \text{where } \lambda := 1 - \mathbf{1}' \nabla H(\mathfrak{L}^{\mathfrak{r}}). \quad (3.1)$$

If  $\mathfrak{p}$  satisfies the integrability condition (2.2), then it is a portfolio, and the following master equation holds:

$$\log \left( \frac{V_T^{\mathfrak{p}}}{V_T^{\mathfrak{r}}} \right) = H(\mathfrak{L}_T^{\mathfrak{r}}) - H(\mathfrak{L}_0^{\mathfrak{r}}) + \int_0^T h_t dt, \quad \forall T > 0, \quad (3.2)$$

$$\text{where } h = \mathfrak{g}_{\mathfrak{p}}^* - \lambda \mathfrak{g}_{\mathfrak{r}}^* - \frac{1}{2} \sum_{i,j=1}^n \mathfrak{a}_{ij}^{\mathfrak{r}} D_{ij}^2 H(\mathfrak{L}^{\mathfrak{r}}). \quad (3.3)$$

*Remark 3.3.* Except for the change to the log representation, the derivation proceeds analogously to the original master equation [6, Theorem 3.1], derivations of which can also be found in [8, 11]. The intermediate equations in the earlier derivations are each individually generalizable to our setting, shown here as Lemmas 3.4 and 3.5. In the special (original) case where  $X$  are the total capitalizations (shares  $\times$  price per share), then renormalizing by the initial values to go from  $X$  to  $\mathfrak{X} := X / \sum_{i=1}^n X_{i,0}$ , and choosing  $\mathfrak{r}$  to be the market portfolio results in

$$\mathfrak{r} = X / \sum_{i=1}^n X_i = \mathfrak{X} \frac{\sum_{i=1}^n X_{i,0}}{\sum_{i=1}^n X_i} = \frac{\mathfrak{X}}{V^{\mathfrak{r}}} = \mathfrak{X}^{\mathfrak{r}}. \quad (3.4)$$

Inserting  $\mathfrak{L} := \log \mathfrak{X}$  and this  $\mathfrak{r}$  into Theorem 3.2 recovers the traditional master equation. In the general setting of this paper, however,  $\mathfrak{X}^{\mathfrak{r}}$  and  $\mathfrak{r}$  are quite distinct.

The following two lemmas will be used in the proof of Theorem 3.2.

**Lemma 3.4.** *For any two portfolios  $\mathbf{p}$  and  $\mathbf{r}$  on  $\mathfrak{X}$ , the following hold*

$$d \log \left( \frac{V^{\mathbf{p}}}{V^{\mathbf{r}}} \right) = \sum_{i=1}^p \mathbf{p}_i d\mathfrak{L}_i^{\mathbf{r}} + \mathfrak{g}_{\mathbf{p}}^* dt, \quad (3.5)$$

$$= \sum_{i=1}^p \mathbf{p}_i \frac{d\mathfrak{X}_i^{\mathbf{r}}}{\mathfrak{X}_i^{\mathbf{r}}} - \frac{1}{2} \mathfrak{a}_{\mathbf{p}\mathbf{p}}^{\mathbf{r}} dt. \quad (3.6)$$

*Proof.* To prove (3.5), by definition

$$\begin{aligned} d\mathfrak{L}_i^{\mathbf{r}} &= d\mathfrak{L}_i - d \log V^{\mathbf{r}}, \\ &= \mathfrak{g}_i dt + \mathfrak{s}_i dW_t - d \log V^{\mathbf{r}}. \end{aligned}$$

Plugging this into the right-hand side of (3.5), we get

$$\begin{aligned} \sum_{i=1}^p \mathbf{p}_i d\mathfrak{L}_i^{\mathbf{r}} + \mathfrak{g}_{\mathbf{p}}^* dt &= \sum_{i=1}^p \mathbf{p}_i (\mathfrak{g}_i dt + \mathfrak{s}_i dW_t) - d \log V^{\mathbf{r}} + \mathfrak{g}_{\mathbf{p}}^* dt, \\ &= d \log V^{\mathbf{p}} - d \log V^{\mathbf{r}}. \end{aligned}$$

To prove (3.6), use

$$\begin{aligned} d\mathfrak{L}_i^{\mathbf{r}} &= \frac{d\mathfrak{X}_i^{\mathbf{r}}}{\mathfrak{X}_i^{\mathbf{r}}} - \frac{1}{2} \frac{d \langle \mathfrak{X}_i^{\mathbf{r}}, \mathfrak{X}_i^{\mathbf{r}} \rangle}{(\mathfrak{X}_i^{\mathbf{r}})^2}, \\ &= \frac{d\mathfrak{X}_i^{\mathbf{r}}}{\mathfrak{X}_i^{\mathbf{r}}} - \frac{1}{2} \mathfrak{a}_{ii}^{\mathbf{r}}. \end{aligned}$$

Plugging this into (3.5) and expanding  $\mathfrak{g}_{\mathbf{p}}^*$  with the numéraire invariance property (2.5) yields

$$\begin{aligned} d \log \left( \frac{V^{\mathbf{p}}}{V^{\mathbf{r}}} \right) &= \sum_{i=1}^m \mathbf{p}_i \left( \frac{d\mathfrak{X}_i^{\mathbf{r}}}{\mathfrak{X}_i^{\mathbf{r}}} - \frac{1}{2} \mathfrak{a}_{ii}^{\mathbf{r}} dt \right) + \frac{1}{2} \left( \sum \mathbf{p}_i \mathfrak{a}_{ii}^{\mathbf{r}} - \mathfrak{a}_{\mathbf{p}\mathbf{p}}^{\mathbf{r}} \right) dt, \\ &= \sum_{i=1}^m \mathbf{p}_i \frac{d\mathfrak{X}_i^{\mathbf{r}}}{\mathfrak{X}_i^{\mathbf{r}}} - \frac{1}{2} \mathfrak{a}_{\mathbf{p}\mathbf{p}}^{\mathbf{r}} dt. \end{aligned}$$

□

**Lemma 3.5.** *For any portfolio  $\mathbf{r}$  on  $\mathfrak{X}$ ,*

$$\begin{aligned} \sum_{i=1}^m \mathbf{r}_i d\mathfrak{L}_i^{\mathbf{r}} &= -\mathfrak{g}_{\mathbf{r}}^* dt, \\ \sum_{i=1}^m \mathbf{r}_i \frac{d\mathfrak{X}_i^{\mathbf{r}}}{\mathfrak{X}_i^{\mathbf{r}}} &= 0. \end{aligned}$$

*Proof.* The first equation follows from (3.5) with  $\mathbf{p} = \mathbf{r}$ . For the second, use (3.6) with  $\mathbf{p} = \mathbf{r}$  and note that  $\mathfrak{a}_{\mathbf{r}\mathbf{r}}^{\mathbf{r}} = 0$  by (2.4). □

Now we prove Theorem 3.2.

*Proof of Theorem 3.2.* Plug in (3.1) for  $\mathbf{p}$  into (3.5) and then use Lemma 3.5 to get

$$\begin{aligned} d \log \left( \frac{V^{\mathbf{p}}}{V^{\mathbf{r}}} \right) &= \sum_{i=1}^m (\lambda \mathbf{r}_i + D_i H(\mathfrak{L}^{\mathbf{r}})) d\mathfrak{L}_i^{\mathbf{r}} + \mathfrak{g}_{\mathbf{p}}^* dt, \\ &= \sum_{i=1}^m D_i H(\mathfrak{L}^{\mathbf{r}}) d\mathfrak{L}_i^{\mathbf{r}} + (\mathfrak{g}_{\mathbf{p}}^* - \lambda \mathfrak{g}_{\mathbf{r}}^*) dt, \end{aligned} \quad (3.7)$$

Expanding  $dH(\mathfrak{L}^{\mathbf{r}})$  gives

$$\begin{aligned} dH(\mathfrak{L}^{\mathbf{r}}) &= \sum_{i=1}^m D_i H(\mathfrak{L}^{\mathbf{r}}) d\mathfrak{L}_i^{\mathbf{r}} + \frac{1}{2} \sum_{i,j=1}^m D_{ij}^2 H(\mathfrak{L}^{\mathbf{r}}) d\mathfrak{L}_i^{\mathbf{r}} d\mathfrak{L}_j^{\mathbf{r}}, \\ &= \sum_{i=1}^m D_i H(\mathfrak{L}^{\mathbf{r}}) d\mathfrak{L}_i^{\mathbf{r}} + \frac{1}{2} \sum_{i,j=1}^m \mathfrak{a}_{ij}^{\mathbf{r}} D_{ij}^2 H(\mathfrak{L}^{\mathbf{r}}) dt. \end{aligned}$$

Plugging this into (3.7) yields

$$d \log \left( \frac{V^{\mathbf{p}}}{V^{\mathbf{r}}} \right) = dH(\mathfrak{L}^{\mathbf{r}}) + \left( \mathfrak{g}_{\mathbf{p}}^* - \lambda \mathfrak{g}_{\mathbf{r}}^* - \frac{1}{2} \sum_{i,j=1}^m \mathfrak{a}_{ij}^{\mathbf{r}} D_{ij}^2 H(\mathfrak{L}^{\mathbf{r}}) \right) dt,$$

proving the theorem.  $\square$

### 3.1.1 Mathematical and intuitive analysis using the master equation

One of the main analytical benefits of the master equation is that it is free of stochastic integrals. When formulas for portfolio returns contain stochastic integrals, then the correct analysis may be counterintuitive, and an incorrect analysis may be intuitively appealing, as in the following example.

**Example 3.6** (Portfolios that are always close need not have returns that are close). Consider two portfolios: Let  $\pi$  be a constant weight portfolio so that  $\pi_t = p \in \mathbb{R}^n$ ,  $\forall t \geq 0$ , and let  $\tilde{\pi}$  be a passive (buy-and-hold) portfolio starting from the same initial allocation  $\tilde{\pi}_0 = p$ . Recalling that the usual description of the wealth process of an arbitrary portfolio  $\theta$  is

$$\log V_T^\theta = \int_0^T \gamma_{\theta,t} dt + \int_0^T \theta_t' \sigma_t dW_t, \quad (3.8)$$

then we may write

$$|\log V_T^\pi - \log V_T^{\tilde{\pi}}| \leq \int_0^T |\gamma_{\pi,t} - \gamma_{\tilde{\pi},t}| dt + \left| \int_0^T (\pi_t - \tilde{\pi}_t)' \sigma_t dW_t \right|.$$

For each  $\varepsilon > 0$  there exists a path  $\omega_\varepsilon$  such that  $\|\tilde{\pi}_t(\omega_\varepsilon) - \pi_t(\omega_\varepsilon)\| < \varepsilon$  for all  $0 \leq t \leq T$ . If  $\gamma$  and  $\sigma$  are bounded, then it is tempting to make the erroneous conclusion that

$$|\log V_T^\pi(\omega_\varepsilon) - \log V_T^{\tilde{\pi}}(\omega_\varepsilon)| \leq \kappa(\varepsilon), \quad (3.9)$$

for some function  $\kappa(\varepsilon)$  satisfying  $\lim_{\varepsilon \rightarrow 0} \kappa(\varepsilon) = 0$ . The erroneous conclusion might be stated in words as:



“If  $\pi$  and  $\tilde{\pi}$  are always nearly identical, then their associated terminal returns at time  $T$  must be nearly identical.”

The erroneous analysis can be avoided by noting that  $\pi$  and  $\tilde{\pi}$  are functionally generated by generating functions  $H(y) = \sum_{i=1}^n p_i y_i$ , and  $\tilde{H}(y) = \log(\sum_{i=1}^n p_i e^{y_i - l_i})$ , respectively, where  $l = L_0 \in \mathbb{R}^n$ . Comparing their wealth processes via their master equations gives the pathwise equation

$$\log V_T^\pi - \log V_T^{\tilde{\pi}} = H(L_T) - H(L_0) - \left( \tilde{H}(L_T) - \tilde{H}(L_0) \right) + \int_0^T \gamma_{p,s}^* ds.$$

If the covariance is uniformly elliptic (there exists  $u > 0$  such that  $y' a_t y \geq u \|y\|^2$ , for all  $y \in \mathbb{R}^n, t \geq 0$ ), and if  $p_i > \delta \in (0, 1), \forall i$ , then  $\gamma_p^* > u \in (0, \infty)$  [11, Lemma 3.4]. If furthermore  $\omega_\varepsilon$  satisfies  $\|L_T(\omega_\varepsilon) - L_0(\omega_\varepsilon)\| < \varepsilon$  (which is actually weaker than the assumption that  $\pi$  and  $\tilde{\pi}_t$  are *always* close), then

$$\log V_T^\pi(\omega_\varepsilon) - \log V_T^{\tilde{\pi}}(\omega_\varepsilon) > \kappa(\varepsilon) + Tu,$$

for some function  $\kappa(\varepsilon)$  satisfying  $\lim_{\varepsilon \rightarrow 0} \kappa(\varepsilon) = 0$ . This correct pathwise analysis is intuitive, and negates the erroneous conclusion of (3.9). Rather: “portfolios that are arbitrarily close at all times need not have returns that are close.”

As exemplified above, the master-equation description of relative return (3.2) has an advantage over the usual description (3.8) for scenario analysis (as described in e.g. [14]). Specifically, it simplifies the analysis of the terminal wealth of an FGP conditional on the terminal values of the underlying synthetic assets. This is because the bulk of an FGP’s performance is typically attributable to the first terms of (3.2), which is entirely determined by the terminal values of the underlying assets. Whereas the second term involves the quadratic variation of the path, and is usually easier to estimate with higher precision. Scenario analysis is more problematic when Itô integrals are present, as in the traditional formulation (3.8).

### 3.1.2 Translation equivariance and numéraire invariance

Generating functions are overspecified in the following sense. Given a generating function  $H$ , each member of the equivalence class of generating functions

$$[H] := \{\kappa + H \mid \kappa \in \mathbb{R}\} \tag{3.10}$$

yields the same function  $\nabla H$ . Hence, given an arbitrary market  $\mathfrak{X}$  and numéraire  $\mathfrak{r}$ , any member of  $[H]$  yields the same functionally generated portfolio (3.1).

**Definition 3.7.** Generating function  $H$  is *translation equivariant* if

$$H(y + \mathbf{1}\kappa) = \kappa + H(y), \quad \forall y \in \mathbb{R}^m.$$

When  $H$  is translation equivariant, then  $H$  and hence the corresponding FGP depends only on the *relative* prices of the synthetic market, not on the *absolute* price level. An example of a class of translation-equivariant

generating functions is the diversity- $p$  family (see [9, 11]):

$$H_p(y) = \frac{1}{p} \log \left( \sum_{i=1}^n \exp\{py_i\} \right), \quad y \in \mathbb{R}^m.$$

The following proposition demonstrates a numéraire-invariance property of the variance capture term  $h$  when  $H$  is translation equivariant.

**Proposition 3.8.** *Let  $H$  be a translation-equivariant generating function satisfying Theorem 3.2, and  $\mathbf{q}$ ,  $\mathbf{u}$ , and  $\mathbf{v}$  be arbitrary portfolios on  $\mathfrak{X}$ . Then*

$$\log \frac{V_T^{\mathbf{p}}}{V_T^{\mathbf{r}}} = H(\mathfrak{L}_T^{\mathbf{r}}) - H(\mathfrak{L}_0^{\mathbf{r}}) + \int_0^T h_t dt, \quad (3.11)$$

where  $\mathbf{p} = \nabla \log H(\mathfrak{L}^{\mathbf{q}})$ ,

$$\text{and } h = \mathfrak{g}_{\mathbf{p}} - \frac{1}{2} \sum_{i,j=1}^m D_{ij}^2 H(\mathfrak{L}^{\mathbf{u}}) \mathbf{a}_{ij}^{\mathbf{v}}. \quad (3.12)$$

*Proof.* Starting with Theorem 3.2, we show that when  $H$  is translation equivariant, then  $\lambda$  of (3.1) is identically 0:

$$\begin{aligned} \frac{\partial}{\partial \kappa} H(y + \mathbf{1}\kappa) &= \frac{\partial}{\partial \kappa} (H(y) + \kappa) = 1, \\ \frac{\partial}{\partial \kappa} H(y + \mathbf{1}\kappa) &= \sum_{i=1}^m \frac{\partial(y_i + \kappa)}{\partial \kappa} D_i H(y + \mathbf{1}\kappa) = \sum_{i=1}^m D_i H(y + \mathbf{1}\kappa) = 1 - \lambda. \end{aligned}$$

Next, we show that  $\mathbf{a}^{\mathbf{r}}$  may be formally replaced with  $\mathbf{a}^{\mathbf{v}}$  in (3.3). We note that

$$\sum_{i=1}^m D_{ij}^2 H(y) = D_j \sum_{i=1}^m D_i H(y) = D_j (1) = 0, \quad \forall y \in \mathbb{R}^m.$$

Using this and the form (2.4) for  $\mathbf{a}^{\mathbf{r}}$  yields

$$\begin{aligned} \sum_{i,j=1}^m D_{ij}^2 H \mathbf{a}_{ij}^{\mathbf{r}} &= \sum_{i,j=1}^m (\mathbf{a}_{ij} - [\mathbf{a}^{\mathbf{r}}]_j - [\mathbf{a}^{\mathbf{r}}]_i - \mathbf{a}_{\mathbf{r}\mathbf{r}}) D_{ij}^2 H, \\ &= \sum_{i,j=1}^m D_{ij}^2 H \mathbf{a}_{ij} - 2 \sum_j [\mathbf{a}^{\mathbf{r}}]_j D_j \sum_{i=1}^m D_i H - \mathbf{a}_{\mathbf{r}\mathbf{r}} \sum_{i,j=1}^m D_{ij}^2 H, \\ &= \sum_{i,j=1}^m D_{ij}^2 H \mathbf{a}_{ij}. \end{aligned}$$

Reversing the steps shows that  $\mathbf{a}$  may be replaced with  $\mathbf{a}^{\mathbf{v}}$  for arbitrary  $\mathbf{v}$ .

It remains to show that  $\mathfrak{L}^{\mathbf{r}}$  may be replaced with  $\mathfrak{L}^{\mathbf{u}}$  as the argument to  $D^2 H$ :

$$D_{ij}^2 H(y + \mathbf{1}\kappa) = D_{ij}^2 (H(y) + \kappa) = D_{ij}^2 H(y), \quad \forall y \in \mathbb{R}^m, \forall \kappa \in \mathbb{R}.$$

Since  $\mathfrak{L}^{\mathfrak{r}} = \mathfrak{L} - \log V^{\mathfrak{r}}$ , this implies that

$$h = \mathfrak{g}_{\mathfrak{p}}^* - \frac{1}{2} \sum_{i,j=1}^m D_{ij}^2 H(\mathfrak{L}^{\mathfrak{r}}) \mathfrak{a}_{ij}^{\mathfrak{v}} = \mathfrak{g}_{\mathfrak{p}}^* - \frac{1}{2} \sum_{i,j=1}^m D_{ij}^2 H(\mathfrak{L}^{\mathfrak{u}}) \mathfrak{a}_{ij}^{\mathfrak{v}}.$$

□

The numéraire  $\mathfrak{r}$  merely sets a stochastic relative price level for  $V^{\mathfrak{p}}$  and  $\mathfrak{L}$  in the master equation. When  $H$  is translation equivariant, then the corresponding FGP and wealth process have no sensitivity to the price level, as can be seen by Proposition 3.8 and

$$H(\mathfrak{L}^{\mathfrak{r}}) = H(\mathfrak{L} - \mathbf{1} \log V^{\mathfrak{r}}) = H(\mathfrak{L}) - \log V^{\mathfrak{r}},$$

which simplifies (3.11) to

$$\log V_T^{\mathfrak{p}} = H(\mathfrak{L}_T) - H(\mathfrak{L}_0) + \int_0^T h_t dt.$$

So, whereas in general each choice of numéraire results in a different master equation, when  $H$  is translation equivariant these different master equations are in fact a trivial translation of the same one equation.

### 3.1.3 Passive numéraires and gauge freedom

The historical work on FGPs [6, 8, 9, 11] takes  $X$  as the total capitalizations,  $\mathfrak{X} = X / \sum_{i=1}^n X_{i,0}$ , and the numéraire  $\mathfrak{r}$  as the market portfolio, leading to  $\mathfrak{r} = \mathfrak{X}^{\mathfrak{r}}$  (see (3.4)), as in Remark 3.3. The important property of the market portfolio that was exploited in those works was its passivity on  $X$  (see Definition 3.9). In the traditional case, the equality of  $\mathfrak{r}$  and  $\mathfrak{X}^{\mathfrak{r}}$  means that  $\sum_{i=1}^n \mathfrak{X}_i^{\mathfrak{r}} = 1$ , hence the generating function need not be defined on all of  $\mathbb{R}^n$ . In this section we explore more generally to what extent a passive numéraire allows a reduction of the domain of the generating function.

**Definition 3.9.** A portfolio  $\mathfrak{r}$  on  $\mathfrak{X}$  is *passive on  $\mathfrak{X}$*  if there exists a constant  $s \in \mathbb{R}^m$ , called the *shares*, such that

$$V^{\mathfrak{r}} = s' \mathfrak{X} \quad \text{and} \quad \mathfrak{r}_i = \frac{s_i \mathfrak{X}_i}{s' \mathfrak{X}}, \quad 1 \leq i \leq m.$$

A portfolio  $\pi$  is *passive* if it is passive on  $X$ .

Passive portfolios are special because they are untraded after the initial allocation, so are unaffected by transaction costs and other liquidity concerns. Generally, the  $\mathfrak{X}_i$  are unbounded from above, so in order that  $V^{\mathfrak{r}} > 0$  is guaranteed, we assume henceforth that any passive portfolio on  $\mathfrak{X}$  is long-only. That is, that  $s \in [0, \infty)^m \setminus \{0\}$ .

When the numéraire  $\mathfrak{r}$  is passive on  $\mathfrak{X}$  then a generating function need not be defined on all of  $\mathbb{R}^m$ , as  $\mathfrak{X}^{\mathfrak{r}}$  will be confined to a hyperplane. Let  $s \in [0, \infty)^m \setminus \{0\}$  be the constant vector of shares such that  $V^{\mathfrak{r}} = s' \mathfrak{X}$ .

Then we have

$$s' \mathfrak{X}^\mathfrak{r} = \frac{s' \mathfrak{X}}{V^\mathfrak{r}} = \sum_{i=1}^m \mathfrak{r}_i = 1.$$

Thus,  $\mathfrak{X}^\mathfrak{r}$  is confined to a hyperplane of codimension 1, and defining a generating function on

$$E_s^m := \{y \in \mathbb{R}^m \mid \sum_{i=1}^m s_i e^{y_i} = 1\}$$

should be sufficient. However, the form of Theorem 3.2 utilizes the Cartesian coordinate system of  $\mathbb{R}^m$ , which is quite convenient, so we sacrifice some generality and require that generating functions be defined on a neighborhood in  $\mathbb{R}^m$  containing  $E_s^m$ .

**Definition 3.10.** Let  $s \in [0, \infty)^m \setminus \{0\}$  and let the passive-on- $\mathfrak{X}$  portfolio  $\mathfrak{r}$  be given by  $\mathfrak{r}_i = s_i \mathfrak{X}_i / s' \mathfrak{X}$ ,  $1 \leq i \leq m$ . An  $\mathfrak{r}$ -generating function is a function  $H \in \mathcal{C}^2(U, \mathbb{R})$ , where  $U$  is a neighborhood in  $\mathbb{R}^m$  containing  $E_s^m$ .

**Gauge freedom** When  $\mathfrak{r}$  is the market portfolio,  $X$  are the total capitalizations, and  $\mathfrak{X} = X / \sum_{i=1}^n X_{i,0}$ , then  $s = \mathbf{1}$ . In [9, Proposition 3.1.14], where generating functions are specified as  $G(x) := \exp\{H(\log x)\}$ , it is shown that in this setting generating functions have the following redundancy: Generating functions  $H_1$  and  $H_2$  generate the same portfolio if and only if  $H_1 - H_2$  is constant on  $E_1^m$ . This generalizes for more general passive portfolios as follows.

**Proposition 3.11.** Let  $\mathfrak{r}$  be passive on  $\mathfrak{X}$  with corresponding shares  $s \in [0, \infty)^m \setminus \{0\}$ . Let  $H_1$  and  $H_2$  be two  $\mathfrak{r}$ -generating functions defined on a neighborhood  $U$  containing  $E_s^m$ . Then  $H_1$  and  $H_2$  generate the same portfolio for any realization of  $\mathfrak{X}$  if and only if  $H_1 - H_2$  is constant on  $E_s^m$ .

*Proof.* Let  $\mathfrak{p}^j$  be the portfolio generated by  $H_j$ ,  $j \in \{1, 2\}$ . The condition  $\mathfrak{p}_i^1 = \mathfrak{p}_i^2$ ,  $1 \leq i \leq m$  for all realizations of  $\mathfrak{X}$  is equivalent by (3.1) to the following holding  $\forall y \in E_s^m$ :

$$\begin{aligned} \left(1 - \sum_{j=1}^m D_j H_1(y)\right) \mathfrak{r}_i + D_i H_1(y) &= \left(1 - \sum_{j=1}^m D_j H_2(y)\right) \mathfrak{r}_i + D_i H_2(y), \quad 1 \leq i \leq m, \\ \iff \frac{s_i e^{y_i}}{\sum_{k=1}^m s_k e^{y_k}} \sum_{j=1}^m D_j (H_1(y) - H_2(y)) &= D_i (H_1(y) - H_2(y)), \quad 1 \leq i \leq m, \\ \iff s_i e^{y_i} \kappa(y) &= D_i (H_1(y) - H_2(y)), \quad 1 \leq i \leq m, \end{aligned} \tag{3.13}$$

for some function  $\kappa$ . Differentiating the equation determining the surface  $E_s^m$  shows that (3.13) is equivalent to  $\nabla(H_1 - H_2)$  being orthogonal to  $E_s^m$ , hence equivalent to  $H_1 - H_2$  being constant on  $E_s^m$ .  $\square$

The gauge freedom implied by Proposition 3.11, specifically by (3.13), is that if  $H$  generates  $\mathfrak{p}$ , then

$$H_f(y) := f \left( \sum_{i=1}^m s_i e^{y_i} \right) + H(y), \quad y \in U \supset E_s^m,$$

also generates  $\mathbf{p}$ , for any  $f \in \mathcal{C}^2((0, \infty), \mathbb{R})$ . This gauge freedom allows one to make any convenient choice for  $f$  in order to simplify calculations. The  $\mathfrak{r}$ -generating functions and general generating functions have their associated respective equivalence classes:

$$[H]_{\mathfrak{r}} := \{f(s'e') + H(\cdot) \mid f \in \mathcal{C}^2((0, \infty), \mathbb{R})\}, \quad \text{where } s_i := \frac{\mathfrak{r}_{i,0}}{\mathfrak{X}_{i,0}}, \quad 1 \leq i \leq m,$$

$$[H] := \{H + \kappa \mid \kappa \in \mathbb{R}\}.$$

Each member of a given  $[H]_{\mathfrak{r}}$  or  $[H]$  is equivalent for the purposes of  $\mathfrak{r}$ -FGPs, or generally FGPs, respectively.

### 3.2 Stochastic generating functions

It is natural to adjust a portfolio based on changing market conditions. However, the FGPs of Theorem 3.2 adjust their weights only as a deterministic function of the underlying relative log price process  $\mathfrak{L}^{\mathfrak{r}}$ , which doesn't allow for much flexibility. Ideally, one would like to be able to modify the generating function stochastically while preserving a useful pathwise description of relative return that is free from stochastic integrals. As a step in this direction, time-dependent generating functions have already been introduced in [9]. In this section we extend that idea to allow a dependence on auxiliary stochastic processes of finite variation.

For  $H : \mathbb{R}^m \times \mathbb{R}^k \rightarrow \mathbb{R}$ , let  $\nabla_l$  denote the gradient with respect to the first ( $m$ -dimensional) argument of  $H$  and let  $\nabla_f$  be the gradient with respect to the second ( $k$ -dimensional) argument.

**Theorem 3.12.** *Let  $H \in C^{2,1}(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R})$  and let  $F$  be an  $\mathbb{R}^k$ -valued process of finite variation having continuous paths. Then the portfolio*

$$\mathbf{p} = \lambda \mathfrak{r} + \nabla_l H(\mathfrak{L}^{\mathfrak{r}}, F), \quad \text{where } \lambda = 1 - \mathbf{1}' \nabla_l H(\mathfrak{L}^{\mathfrak{r}}, F),$$

*satisfies the following master equation:*

$$\log \left( \frac{V_T^{\mathbf{p}}}{V_T^{\mathfrak{r}}} \right) = H(\mathfrak{L}_T^{\mathfrak{r}}, F_T) - H(\mathfrak{L}_0^{\mathfrak{r}}, F_0) - \int_0^T [\nabla_f H(\mathfrak{L}_t^{\mathfrak{r}}, F_t)]' dF_t + \int_0^T h_t dt, \quad (3.14)$$

$$\text{where } h = \mathfrak{g}_{\mathbf{p}}^* - \lambda \mathfrak{g}_{\mathfrak{r}}^* - \frac{1}{2} \sum_{i,j=1}^m D_{l_i l_j}^2 H(\mathfrak{L}^{\mathfrak{r}}, F) \mathfrak{a}_{ij}^{\mathfrak{r}}.$$

*Proof.* Since  $F$  has continuous paths and is finite variation, Itô's formula yields

$$dH(\mathfrak{L}^{\mathfrak{r}}, F) = (d\mathfrak{L}^{\mathfrak{r}})' \nabla_l H(\mathfrak{L}^{\mathfrak{r}}, F) + (dF)' \nabla_f H(\mathfrak{L}^{\mathfrak{r}}, F) + \frac{1}{2} \sum_{i,j=1}^m d\mathfrak{L}_i^{\mathfrak{r}} d\mathfrak{L}_j^{\mathfrak{r}} D_{l_i l_j}^2 H(\mathfrak{L}^{\mathfrak{r}}, F).$$

The rest of the proof is straightforward from the proof of Theorem 3.2. □

While a stochastic FGP of the type of Theorem 3.12 loses some elegance and tractability in its master equation, the extra flexibility gained can be useful in practice. For example,  $F$  may be factors that inform portfolio construction, such as those of Fama and French [3, 4], fundamental economic data such as bond yields or stock market diversity [7, 9, 11, 12], or information extracted from Twitter feeds [2].

It is possible to remove the restrictions on  $F$  - that it is finite variation and has continuous paths - to derive a more general master equation, but this would make the correction term  $\int_0^T [\nabla_f H(\mathfrak{L}_t^r, F_t)]' dF_t$  of (3.14) more complex. A continuous  $F$  of finite variation is sufficient for the applications that follow, so we do not pursue these extensions here.

The following corollary specializes Theorem 3.12 to the case where  $F$  is the log of a stochastic discount factor for  $\mathfrak{X}^r$ .

**Corollary 3.13.** *Let  $H \in C^2(\mathbb{R}^m, \mathbb{R})$ ,  $F$  be an  $\mathbb{R}^m$ -valued process of finite variation, and  $\mathfrak{Z}^r := \mathfrak{L}^r - F$ . Then the portfolio*

$$\mathfrak{p} = \lambda \mathfrak{r} + \nabla H(\mathfrak{Z}^r), \quad \text{where } \lambda = 1 - \mathbf{1}' \nabla H(\mathfrak{Z}^r),$$

*satisfies the following master equation:*

$$\begin{aligned} \log \left( \frac{V_T^p}{V_T^r} \right) &= H(\mathfrak{Z}_T^r) - H(\mathfrak{Z}_0^r) + \int_0^T h_t dt + \int_0^T [\mathfrak{p}_t - \lambda_t \mathfrak{r}_t]' dF_t, \\ \text{where } h &= \mathfrak{g}_p^* - \lambda \mathfrak{g}_r^* - \frac{1}{2} \sum_{i,j=1}^m D_{ij}^2 H(\mathfrak{Z}^r) \mathfrak{a}_{ij}^r. \end{aligned}$$

*Proof.* Let  $Q(l, f) := H(l - f)$ . Then Theorem 3.12 applied to generating function  $Q$  yields that  $\mathfrak{p}$  and  $\lambda$  given above obey the master equation

$$\begin{aligned} \log \left( \frac{V_T^p}{V_T^r} \right) &= H(\mathfrak{Z}_T^r) - H(\mathfrak{Z}_0^r) + \int_0^T h_t dt + \int_0^T (\nabla H(\mathfrak{Z}_t^r))' dF_t, \\ h &= \mathfrak{g}_p^* - \lambda \mathfrak{g}_r^* - \frac{1}{2} \sum_{i,j=1}^m D_{ij}^2 H(\mathfrak{Z}^r) \mathfrak{a}_{ij}^r. \end{aligned}$$

Plugging in  $\nabla H(\mathfrak{Z}^r) = \mathfrak{p} - \lambda \mathfrak{r}$  completes the proof.  $\square$

When  $H$  is chosen to be translation equivariant in its first argument, then  $\lambda = 0$ , as in Proposition 3.8, and the correction term simplifies to the integrated total exposure of  $\mathfrak{p}$  to  $F$ .

Stochastic discount factors can be useful in cases where it is more natural to prescribe the generating function on the log of discounted prices  $\mathfrak{Z}$  than on the log prices  $\mathfrak{L}$ .

The following example looks at the strategy of switching from an initial FGP to a subsequent one at a stopping time. The overall portfolio is an example of a stochastic FGP.

**Example 3.14** (Stochastic switching between FGPs<sup>1</sup>). For generating functions  $H_1$  and  $H_2$ , and stopping time  $\tau$ , the generating function

$$H(y, i) := iH_1(y) + (1 - i)H_2(y), \quad y \in \mathbb{R}^n, \ i \in \mathbb{R},$$

will generate an FGP that switches between  $\mathfrak{p}^{(1)}$  generated by  $H_1$  and  $\mathfrak{p}^{(2)}$  generated from  $H_2$  at  $\tau$ , when  $\mathbf{1}_{t \leq \tau}$  is supplied as the stochastic second argument. This type of portfolio was used by Banner and D. Fernholz in [1] for constructing arbitrages relative to the market portfolio at arbitrarily short deterministic

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<sup>1</sup>The author wishes to thank Radka Pickova for suggesting this idea.

horizons  $T > 0$  in a class of market models including volatility-stabilized markets [10]. Those models have also been shown to admit functionally generated relative arbitrage over sufficiently long time horizons [10]. However, there is a horizon before which functionally generated relative arbitrage is not possible, regardless of the choice of generating function [17]. This example shows that relative arbitrages exist on arbitrarily short horizons within the class of FGPs having a *stochastic* generating function.

## 4 Applications

### 4.1 Long-short statistical arbitrage with FGPs

The paper [5] of R. Fernholz and C. Maguire introduces an idea for a statistical arbitrage strategy in markets where the realized rate of variance of log market prices depends on the sampling interval. The general idea is to take a long position in an FGP that is rebalanced over a time interval corresponding to a high variance rate, hedged with a short position in an FGP generated from the *same* generating function, but rebalanced over a *different* time interval corresponding to a low variance rate. Statistical arbitrage profits accrue from the different rates of variance-capture, as represented by different  $h$  terms in (3.2), since these terms are directly proportional to the variance rate. Because the long and short FGPs have the same generating function, their corresponding  $H$  terms of (3.2) are identical, providing an effective hedge for each other.

The data presented in [5] indicate that for 2005, the variance rate was significantly higher at higher sampling frequencies intradaily for large-cap US equities. The authors looked at rebalancing the long component at 90-second intervals and rebalancing the short component once a day. These choices of rebalancing intervals were ad hoc, not the output of an optimization problem. In this section we develop general performance formulas for such long-short statistical arbitrages, creating a framework for optimizing the selection of generating function and rebalancing intervals. The ideas here are closely related to those in [12, Section 8], arising in the context of mirror portfolios (see also Section 4.3 of this paper).

We will show that the growth rate of the statistical arbitrage portfolio always has the quadratic form

$$\gamma_\pi = A\kappa - B\kappa^2, \quad A, B > 0, \quad (4.1)$$

where  $\kappa$  is the leverage factor, that is, the weight invested in the long portfolio. Hence, there is a level of leverage  $\bar{\kappa} = A/B$  above which the portfolio tends to shrink in value rather than grow. The leverage  $\tilde{\kappa} = A/(2B) = \bar{\kappa}/2$  gives the maximal growth rate of  $\tilde{\gamma}_\pi = A^2/(4B)$ .

To estimate the performance of the strategy described in [5], one can treat it as a constant-weight functionally generated portfolio, where the effective market is given by  $\mathfrak{X} := (\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3)$ , with  $\mathfrak{X}_1$  as the value of the portfolio to be held long,  $\mathfrak{X}_2$  as the value of the portfolio to be shorted, and  $\mathfrak{X}_3$  as the money market. The overall portfolio  $\mathbf{p}$  has  $\mathbf{p}_1 = -\mathbf{p}_2 = \kappa \in (0, \infty)$ , with  $\mathbf{p}_3 = 1$ . This portfolio is functionally generated by the generating function  $H(y) = \kappa(y_1 - y_2)$ , so Theorem 3.2 yields

$$\log \hat{V}_T^{\mathbf{p}} = \kappa \left( \Delta_T \hat{\mathfrak{L}}_1 - \Delta_T \hat{\mathfrak{L}}_2 \right) + \int_0^T h_s ds.$$

Although they are actually discretely traded,  $\mathfrak{X}_1$  and  $\mathfrak{X}_2$  are approximated as the values of continuously

traded FGPs on  $X$ , starting from  $\mathfrak{X}_{1,0} = \mathfrak{X}_{2,0}$ . This approximation has been shown to be accurate for diversity- and entropy-weighted FGP approximations that are rebalanced merely once a month [9, Chapter 6].  $\mathfrak{X}_1$  and  $\mathfrak{X}_2$  have the same generating functions,  $H_1 = H_2$ , so in this approximation their values differ only through their variance-capture terms,  $h_1, h_2$ , which differ only because the rebalancing occurs at different frequencies. Hence, we obtain

$$\log \hat{V}_T^p \approx \kappa \left( \int_0^T h_{1,s} ds - \int_0^T h_{2,s} ds \right) + \int_0^T h_s ds.$$

Constant-weight FGPs are generated by linear  $H$  and yield  $h = \gamma_p^*$  from (3.2). Therefore,

$$\log \hat{V}_T^p \approx \int_0^T \left[ \kappa \left( h_{1,s} - h_{2,s} + \frac{a_{11} - a_{22}}{2} \right) - \frac{\kappa^2}{2} a_{(1,-1)(1,-1)} \right] ds.$$

This has the quadratic form of (4.1). To estimate the parameters, we approximate the stochastic quantities as constant, and plug in the value of their sample estimators. We can identify

$$\begin{aligned} A &= h_1 - h_2 + \frac{1}{2}(a_{11} - a_{22}), \\ B &= \frac{1}{2}a_{(1,-1)(1,-1)}. \end{aligned}$$

In [5] the FGP chosen was the equal-weight portfolio on large cap US equities, specifically those in the S&P500 and/or Russell 1000 in 2005. The annualized sample averages for that year were  $a_{11} = .0683$ ,  $a_{22} = .0423$ ,  $a_{(1,-1)(1,-1)} = 1.69 \times 10^{-7}$ ,  $h_1 = 0.0341$ ,  $h_2 = 0.0211$ .<sup>2</sup> These result in  $A = 0.0260$ ,  $B = 8.45 \times 10^{-8}$ ,  $\tilde{\kappa} = \frac{A}{2B} = 1.54 \times 10^5$ , and  $\tilde{\gamma}_\pi = 2.00 \times 10^3$ .

While  $\tilde{\kappa}$  is not of the order of magnitude usually seen in portfolio construction, it must be remembered that it is not a weight for investment into equities, but rather into a long-short combination of two very diverse portfolios that are *very similar* nearly always. At the beginning of each day, the long and short portfolios are equal, so the net position in each equity starts at 0. Each FGP is an equal-weight portfolio in about 1000 equities. If we approximate each initial weight as  $10^{-3}$  and use a leverage factor of  $\kappa = 1.5 \times 10^5$ , then an isolated 1% intraday price movement of a particular equity induces a change in net weight of 1.5 in that equity. While this is still an unrealistically leveraged portfolio, it is much closer to a reasonable order of magnitude considering that it would be offset by similarly sized positions of opposite sign.

Despite the above remark, the amount of leverage involved in the  $\tilde{\kappa}$  portfolio is prohibitive due to the realities of equity markets that lie outside of the framework of this paper, such as price jumps, margin requirements, transaction costs, short-selling fees, liquidity constraints, etc. It is these factors then that become the limiting ones for the level of leverage to use in seeking profitability from a statistical arbitrage portfolio of this type. A more plausible level of leverage of  $\kappa = 1 \times 10^3$  results in  $\gamma_\pi = 26$ , still orders of magnitude outside the realm of documented performance.

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<sup>2</sup>These are the numbers from [5], after transforming standard deviations to variances and annualizing all numbers:  $a_{11} = 250 * 0.000273$ ,  $a_{22} = 250 * 0.000169$ ,  $a_{(1,-1)(1,-1)} = 250 * \left(\frac{0.0026}{100}\right)^2$ ,  $h_1 = 0.0341$ ,  $h_2 = 0.0211$ .



## 4.2 Immunization

Suppose that we have selected a generating function that is otherwise appealing, except that it produces a generated portfolio that is exposed, relative to the numéraire, to risk factors that the investor would rather remain unexposed to. For example, the investor may wish to avoid taking on numéraire risk by maintaining zero excess exposure to it.

One way to remove unwanted risk exposure is to modify the initial generating function, only in so far as to make it invariant to changes in the argument along the direction of given risk factors. To be more concrete, suppose that  $H$  is the initial generating function, and that  $\beta^1, \dots, \beta^K$  are each continuous-path finite variation processes in  $\mathbb{R}^m$  satisfying

$$(\beta_t^k)' \beta_t^j = \delta_{kj}, \quad 1 \leq k, j \leq K, \quad \forall t \geq 0.$$

The orthonormal set of random vectors  $\{\beta_t^1, \dots, \beta_t^K\}$  spans the subspace in  $\mathbb{R}^m$  that we would like to immunize the generated portfolio's performance to at time  $t$ . That is, we would like to find a generating function  $\tilde{H}$ , similar to  $H$ , except also obeying

$$(\beta_t^k)' \nabla \tilde{H}_t = 0, \quad \forall t \geq 0.$$

For this to hold,  $\tilde{H}$  will need to be stochastic, taking  $\beta := \{\beta_i^k\}_{1 \leq i \leq m}^{1 \leq k \leq K}$  as a second argument. Perhaps the most natural way to modify  $H$  in order to achieve this is to project  $\nabla H$  onto the complement of the span of  $\{\beta^1, \dots, \beta^K\}$  and allow that to determine the new function  $\tilde{H}$ . To that end, let  $P^\perp(y, b)$  be the projection operator that projects  $y$  onto the orthogonal complement of the span of vectors  $\{b^k\}^{1 \leq k \leq K}$ . It is simplest to specify  $P^\perp$  in the case where  $\{b^k\}$  are orthonormal:

$$P^\perp(y, b) = y - \sum_{k=1}^K (y' b^k) b^k, \quad \text{for } (b^k)' b^j = \delta_{kj}, \quad 1 \leq k, j \leq K.$$

**Proposition 4.1.** *Let  $\{\beta^1, \dots, \beta^K\}$  be  $K \leq m$  finite variation processes in  $\mathbb{R}^m$  that are mutually orthonormal at all times. The generating function  $H$  and generated portfolio*

$$\mathbf{p} = \lambda \mathbf{r} + P^\perp(\nabla H(P^\perp(\mathfrak{L}^\mathbf{r}, \beta)), \beta), \quad \text{where } \lambda = 1 - \mathbf{1}' P^\perp(\nabla H(P^\perp(\mathfrak{L}^\mathbf{r}, \beta)), \beta),$$

*satisfy the following master equation:*

$$\log \left( \frac{V_T^\mathbf{p}}{V_T^\mathbf{r}} \right) = H(P^\perp(\mathfrak{L}_T^\mathbf{r}, \beta_T)) - H(P^\perp(\mathfrak{L}_0^\mathbf{r}, \beta_0)) + \int_0^T h_t dt - \sum_{k=1}^K \int_0^T [\nabla_{b^k} H(P^\perp(\mathfrak{L}_t^\mathbf{r}, \beta_t))] d\beta_t^k,$$

where

$$\begin{aligned}
h &= \mathbf{g}_p^* - \lambda \mathbf{g}_r^* - \frac{1}{2} \left( \sum_{i,j=1}^m \mathbf{a}_{ij}^r D_{ij}^2 H(P^\perp(\mathfrak{L}^r, \beta)), \beta \right) - 2 \sum_{k=1}^K (b^k)' \mathbf{a}^r D^2 H(P^\perp(\mathfrak{L}^r, \beta)) b^k \\
&\quad + \sum_{k=1}^K \sum_{k'=1}^K (b^{k'})' \mathbf{a}^r b^k (b^{k'})' D^2 H(P^\perp(\mathfrak{L}^r, \beta)) b^k, \\
\nabla_{b^k} H(P^\perp(y, b)) &= - \left[ (b^k)' \nabla H(P^\perp(y, b)) \right] y_i - \left[ (b^k)' y \right] D_i H(P^\perp(y, b)).
\end{aligned}$$

*Proof.* Define

$$\begin{aligned}
\tilde{H} : \mathbb{R}^m \times \mathbb{R}^{K \times m} &\rightarrow \mathbb{R}, \\
\tilde{H}(y, b) &= H(P^\perp(y, b)), \\
\text{where } P^\perp(y, b) &= y - \sum_{k=1}^K (y' b^k) b^k.
\end{aligned}$$

The result is then obtained from a straightforward application of Theorem 3.12 to  $\tilde{H}$ . The relevant derivatives are

$$\begin{aligned}
\nabla_y \tilde{H}(y, b) &= \nabla H(P^\perp(y, b)) - \sum_{k=1}^K \left[ (b^k)' \nabla H(P^\perp(y, b)) \right] b^k, \\
&= P^\perp(\nabla H(P^\perp(y, b)), b), \\
\frac{\partial^2}{\partial y_i \partial y_j} \tilde{H}(y, b) &= D_{ij}^2 H(P^\perp(y, b)) - \sum_{k=1}^K b_j^k \left[ (b^k)' D^2 H(P^\perp(y, b)) \right]_i - \sum_{k=1}^K b_i^k \left[ (b^k)' D^2 H(P^\perp(y, b)) \right]_j \\
&\quad + \sum_{k=1}^K \sum_{k'=1}^K b_i^k b_j^{k'} (b^{k'})' D^2 H(P^\perp(y, b)) b^k, \\
\frac{\partial}{\partial b_i^k} \tilde{H}(y, b) &= - \left( (b^k)' \nabla H(P^\perp(y, b)) \right) y_i - \left( (b^k)' y \right) D_i H(P^\perp(y, b)).
\end{aligned}$$

□

The characterization of the performance of the immunized FGP given by Proposition 4.1 is not so pretty, but the idea of what has changed from the non-immunized FGP is straightforward. The relative wealth process  $\log(V^p/V^r)$  of the generated portfolio of Proposition 4.1 is locally not exposed to changes in  $\mathfrak{L}^r$  along the linear span of  $\{\beta^1, \dots, \beta^K\}$ . This can be seen from

$$\begin{aligned}
(\beta^k)' \nabla_y H(P^\perp(y, b)) |_{(\mathfrak{L}^r, \beta)} &= (\beta^k)' (\mathbf{p} - \lambda \mathbf{r}), \\
&= (\beta^k)' P^\perp(\nabla H(P^\perp(\mathfrak{L}^r, \beta)), \beta), \\
&= 0, \quad 1 \leq k \leq K.
\end{aligned}$$

**Example 4.2** (Numéraire exposure). Consider the case where immunization is desired with respect to relative exposure to the numéraire. The appropriate  $\beta$  to use to hedge against excess numéraire exposure is

1 less than the “CAPM  $\beta$ ” of [15]. The instantaneous version of this parameter is

$$\begin{aligned}\tilde{\beta}_{i,t} &= \frac{\frac{d}{dt} \langle \log V^{\mathbf{r}}, \mathfrak{L}_i^{\mathbf{r}} \rangle}{\frac{d}{dt} \langle \log V^{\mathbf{r}} \rangle} = \frac{[\mathbf{a}_t \mathbf{r}_t]_i - \mathbf{a}_{\mathbf{r}\mathbf{r}}}{\mathbf{a}_{\mathbf{r}\mathbf{r}}} = \frac{[\mathbf{a}_t \mathbf{r}_t]_i}{\mathbf{a}_{\mathbf{r}\mathbf{r}}} - 1, \\ &= \beta_{i,t}^{(\text{CAPM})} - 1.\end{aligned}$$

We normalize this to

$$\beta := \tilde{\beta} / \|\tilde{\beta}\|.$$

Although theoretically this instantaneous  $\beta$  may not be a continuous-path finite variation process, in practice the instantaneous  $\beta$  is not observable, and  $\beta$  is typically estimated by time-averaging over some historical time window. The practical *and* theoretical result of such a time-averaging procedure is a continuous-path finite variation process. For example, the estimator might have the theoretical form

$$\begin{aligned}\tilde{\beta}_{i,t} &= \frac{\frac{1}{\Delta t} \int_{t-\Delta t}^t [\mathbf{a}_s \mathbf{r}_s]_i ds}{\frac{1}{\Delta t} \int_{t-\Delta t}^t \mathbf{a}_{\mathbf{r}\mathbf{r},s} ds} - 1, \\ \beta &:= \tilde{\beta} / \|\tilde{\beta}\|,\end{aligned}$$

for some  $\Delta t > 0$ . In practice the integrals are approximated by sums of discretely sampled values.

**Example 4.3** (Price level). Another possibly desirable immunization is to hedge out any exposure to a rise or fall in the overall price level. This can be done by choosing the constant vector  $\beta = m^{-1/2} \mathbf{1}$ .

### 4.3 Mirror portfolios

In this section we use generating functions to elaborate on some of the properties of *mirror portfolios*, introduced in [12]. Mirror portfolios are functionally generated portfolios on their synthetic market  $\mathfrak{X} = (V^{\mathbf{r}}, V^{\mathbf{p}})$ , where  $\mathbf{p}$  is the portfolio to be reflected and  $\mathbf{r}$  is the portfolio serving as the “mirror”.

**Definition 4.4.** If  $\mathbf{p}$  and  $\mathbf{r}$  are portfolios, then the portfolio

$$\tilde{\mathbf{p}}^{[q],\mathbf{r}} := q\mathbf{p} + (1 - q)\mathbf{r}, \quad q \in \mathbb{R},$$

is called the *q-mirror of  $\mathbf{p}$  with respect to  $\mathbf{r}$* . When  $\mathbf{r}$  is fully invested in the money market, then the portfolio is called simply  $\tilde{\mathbf{p}}^{[q]}$ , the *q-mirror of  $\mathbf{p}$* . For  $q = -1$ ,  $\tilde{\mathbf{p}}^{\mathbf{r}} := \tilde{\mathbf{p}}^{[-1],\mathbf{r}}$  is called simply the *mirror of  $\mathbf{p}$  with respect to  $\mathbf{r}$* .

If  $\mathbf{p}$  and  $\mathbf{r}$  are portfolios, then the *q-mirror of  $\mathbf{p}$  with respect to  $\mathbf{r}$*  satisfies the integrability condition (2.2), so is also a portfolio. As an example, let  $X_1$  be the money market, then the portfolio  $e_i := (0, \dots, 0, 1, 0, \dots)$  has the mirror  $\tilde{e}_i = (2, 0, \dots, 0, -1, 0, \dots)$ .

**Proposition 4.5.** *The q-mirror of  $\mathbf{p}$  with respect to  $\mathbf{r}$  is functionally generated from the market  $\mathfrak{X} := (V^{\mathbf{r}}, V^{\mathbf{p}})$*

by the generating function  $H(y_1, y_2) := (1 - q)y_1 + qy_2$ , and thus satisfies

$$\begin{aligned}\log V^{\tilde{\mathbf{p}}^{[q], \mathfrak{r}}} &= (1 - q) \log V^{\mathfrak{r}} + q \log V^{\mathbf{p}} + \int h_s ds, \\ \log \left( \frac{V^{\tilde{\mathbf{p}}^{[q], \mathfrak{r}}}}{V^{\mathfrak{r}}} \right) &= q \log \left( \frac{V^{\mathbf{p}}}{V^{\mathfrak{r}}} \right) + \int h_s ds, \\ \text{where } h &= \mathfrak{g}_{\tilde{\mathbf{p}}^{[q], \mathfrak{r}}}^* = \frac{1}{2} \left( (1 - q)\mathbf{a}_{11} + q\mathbf{a}_{22} - \left[ (1 - q)^2 \mathbf{a}_{11} + 2q(1 - q)\mathbf{a}_{12} + q^2 \mathbf{a}_{22} \right] \right).\end{aligned}$$

If, additionally,  $\mathfrak{r}$  is the money market, then

$$\log \hat{V}^{\tilde{\mathbf{p}}^{[q]}} = q \log \hat{V}^{\mathbf{p}} + \frac{q(1 - q)}{2} \langle \log V^{\mathbf{p}} \rangle.$$

If, additionally,  $q = -1$ , then

$$\log \hat{V}^{\tilde{\mathbf{p}}} = -\log \hat{V}^{\mathbf{p}} - \langle \log V^{\mathbf{p}} \rangle, \quad (4.2)$$

*Proof.* The generating function  $H$  is translation equivariant and  $D^2 H = 0$ , so we may apply Proposition 3.8 to obtain the first result. The others are easy consequences of plugging in  $\mathbf{a}_{11} = \mathbf{a}_{12} = 0$  in the case where  $\mathfrak{r}$  is the money market.  $\square$

The following corollary shows under certain conditions that either a given portfolio  $\mathbf{p}$  or its mirror  $\tilde{\mathbf{p}}$  or both will asymptotically lose all of its wealth. The sufficient conditions are typically satisfied by long-term portfolios of interest in market models of practical relevance.

**Corollary 4.6.** *Suppose that both of the following hold:*

(i)

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \langle \log V^{\mathbf{p}} \rangle_t > 0, \quad a.s.$$

(ii)

$$\lim_{t \rightarrow \infty} \left( \frac{\log \log t}{t^2} \langle \log V^{\mathbf{p}} \rangle_t \right) = 0, \quad a.s. \quad (4.3)$$

Then

$$P \left( \left\{ \lim_{t \rightarrow \infty} \hat{V}_t^{\mathbf{p}} = 0 \right\} \cup \left\{ \lim_{t \rightarrow \infty} \hat{V}_t^{\tilde{\mathbf{p}}} = 0 \right\} \right) = 1.$$

*Proof.* Under (4.3) the law of the iterated logarithm [13, p. 112] implies that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \left( \log \hat{V}_t^{\mathbf{p}} - \int_0^t \mathfrak{g}_{\mathbf{p}, s} ds \right) = 0, \quad a.s.,$$

since the process inside the parentheses is a continuous local martingale. Thus using (4.2), we get

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \frac{1}{t} \left( \log \hat{V}_t^{\tilde{p}} + \langle \log V^p \rangle_t + \int_0^t \mathfrak{g}_{p,s} ds \right) = 0, \quad \text{a.s.}, \\
& \Rightarrow \lim_{t \rightarrow \infty} \frac{1}{t} \left( \log \hat{V}_t^p + \log \hat{V}_t^{\tilde{p}} + \langle \log V^p \rangle_t \right) = 0, \quad \text{a.s.}, \\
& \quad \Rightarrow \lim_{t \rightarrow \infty} \frac{1}{t} \left( \log \hat{V}_t^p + \log \hat{V}_t^{\tilde{p}} \right) < 0, \quad \text{a.s.}, \\
& \Rightarrow P \left( \left\{ \lim_{t \rightarrow \infty} \frac{1}{t} \log \hat{V}_t^p < 0 \right\} \cup \left\{ \lim_{t \rightarrow \infty} \frac{1}{t} \log \hat{V}_t^{\tilde{p}} < 0 \right\} \right) = 1. \quad \square
\end{aligned}$$

Equation (4.2) shows that at least one and possibly both of  $\log \hat{V}^p$  and  $\log \hat{V}^{\tilde{p}}$  have negative drift at any time when  $\langle \log V^p \rangle$  is increasing. The preceding corollary shows that either the long or short position (or both) in a particular portfolio must lose all wealth relative to the money market asymptotically, assuming that the asymptotic local variance rate does not approach 0. A portfolio whose wealth tends to 0 asymptotically would typically be considered a poor long-term investment. We are confronted with the counterintuitive result that in some cases “mirroring” a poor investment is still a poor investment.

## 5 Concluding remarks

The key analytical benefit of portfolios that are functionally generated is the representation of their return relative to a numéraire via a pathwise master equation free of stochastic integrals. The generalizations of FGPs presented here expand the class of portfolio-numéraire pairs that may be analyzed in this way. The dynamism of FGPs is enhanced by the freedom to incorporate processes having continuous, finite-variation paths as auxiliary arguments to generating functions. This allows FGPs to be sensitive to changing market conditions beyond the price changes of the assets. The main applications that we showed are towards statistical arbitrage, portfolio immunization, and the theory of mirror portfolios.

It is a shortcoming of this work that transaction costs are ignored throughout. They are especially important to the performance of the statistical arbitrage portfolios examined in Section 4.1. The inclusion of transaction costs in a tractable way for FGPs in  $n$ -asset markets is a topic of ongoing research. Due to its complexity, it warrants a separate paper that the author hopes will be forthcoming in the future.

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